

EXERCISES

Mathematica 6 ~ Lab Number 3

Problem 1. TrigExpand the functions $\tan 2\theta$ and $\cos 6\theta$. Then **Simplify** your results.

Problem 2. TrigReduce the functions $\sin^2 \theta$ and $\tan^2 \frac{1}{2}\theta$. Then **Simplify** your results.

Problem 3. “Magic squares” have fascinated mathematicians for many centuries. The following example

$$\mathbb{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

is taken from an engraving by Albrecht Dürer. Create a link to the Mathworld website that discusses Dürer’s magic square.

\mathbb{M} displays all the integers from 1 through 16, and has the “magical” property that

$$\sum_{\text{any row}} = \sum_{\text{any column}} = \sum_{\text{counter diagonal}} = \sum_{\text{principal diagonal}} \equiv \text{trace}$$

Ask *Mathematica* to evaluate

- 1) $\text{tr} \mathbb{M}$
- 2) $\det \mathbb{M}$
- 3) the eigenvalues of \mathbb{M}
- 4) the eigenvectors of \mathbb{M}

Notice that you had no reason to expect the eigenvalues to real, but that they turned out “magically” to be so.

Let the eigenvectors (which *Mathematica* has presented as lists) be called a , b , c and d . Evaluate each of the following ten dot products (command **a.a**, etc.):

$$\begin{array}{cccc} a \cdot a & a \cdot b & a \cdot c & a \cdot d \\ & b \cdot b & b \cdot c & b \cdot d \\ & & c \cdot c & c \cdot d \\ & & & d \cdot d \end{array}$$

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What do you conclude about the relation of d to a , b and c ?

Look finally to the validity of each of the four claims that

$$(\text{matrix})(\text{eigenvector}) = (\text{eigenvalue})(\text{eigenvector})$$

Problem 4. Here is a “Latin square” (each row and each column presents a permutation of $\{1, 2, 3, 4\}$):

$$\mathbb{L}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Create a link to the Wikipedia website that discusses Latin squares.

Evaluate $\det \mathbb{L}_1$ and list the eigenvalues of \mathbb{L}_1 .

Problem 5. The following Latin square

$$\mathbb{L}_2 = \begin{pmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 2 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

is distinguished by (among other properties) its symmetry about the principal diagonal. . . on which grounds we are assured that the eigenvalues must be real. Evaluate $\det \mathbb{L}_2$ and list the eigenvalues of \mathbb{L}_2 . Use **N[%]** to list approximate *numerical values* of the eigenvalues (which *Mathematica* prefers to give exactly, when it can).

Problem 6. Now introduce (in the $_{22}$ place) one small symmetry-preserving typo into the description of the preceding matrix, writing

$$\mathbb{L}_3 = \begin{pmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 3 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

Evaluate $\det \mathbb{L}_3$ and list the eigenvalues of \mathbb{L}_3 ; you find that *Mathematica*, confronted with the solution of a quintic, responds unhelpfully to the latter command. See how *Mathematica* responds now to the command **N[%]**

Compare those results with the results produced by the commands

CharacteristicPolynomial[L₃,x]
spectrum=x/.NSolve[%==0,x]

Notice that this little exercise—mere child’s play for *Mathematica*—involves labor you certainly would not want to undertake by hand!

with (say) $A = 3$ and $B = -2$. Proceed as before: plot the spectrum, and compare it with the original spectrum.

The results just obtained would have been much more revealing—more valuable as “toy solid state physics”—if we had assumed the crystal to contain not just 16 “atoms” but (say) 16,000. But the keyboard labor to describe a 16000×16000 matrix to *Mathematica* requires a higher level of technique than the one to which we presently aspire.

Problem 8. The physical question: *What is the gross spin angular momentum of the sun, and how does it compare to what would be the angular momentum of an equivalent number of stationary protons if each contributed a “proton spin” given by $\frac{1}{2}\hbar$ (where, by universal convention, $\hbar \equiv h/2\pi$)?* To approach the problem you will find it convenient to install a couple of standard packages, which is accomplished by commanding

Needs["PhysicalConstants`"]

Needs["Units`"]

To gain some sense of the resources now at your command, you might ask *Mathematica* about

?SolarMass (use **Convert** to convert to kilograms)

?SolarRadius

?ProtonMass

?PlanckConstant

and accept as given that the solar rotational period is 25.36 days. Use that information to compute (i) the angular momentum of the sun, assumed to rotate as a homogeneous solid sphere; (ii) the equivalent number of protons; (iii) the spin angular momentum (at $\frac{1}{2}\hbar$ /each) of such a population, assuming (preposterously!) that all the spins are aligned; (iv) the ratio (latter/former). Be sure to display your final answer as a dimensionless number.